Lecture 5:
Theory of Portfolio Choice

- Introduction
- Diversification
  - Portfolio μ and σ
  - Systematic vs. Non-systematic Risk
- Efficient Frontier
- Optimal Portfolio

Introduction

- Investors typically invest in a *portfolio*, i.e. they hold a collection of different assets.
- By forming a portfolio and spreading your investment over a variety of assets, you can lower the overall level of risk exposure.
- That is, a portfolio of carefully selected assets allows you to *diversify* your investment risk.

Introduction

- An obvious question is:

  How do we select what assets to include in a portfolio and in what combination?

- Specifically, we want to know

  Given a set of available assets (their prices and expected distribution of returns), what is the *optimal portfolio*?

- The Theory of Portfolio Choice is developed to answer the above questions.

Introduction

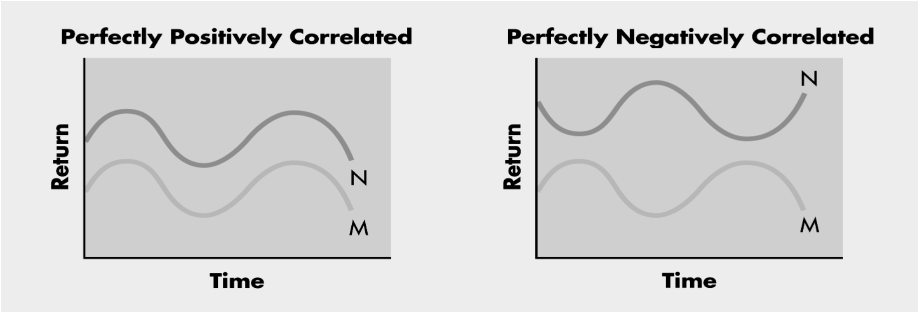
- Before we introduce the Theory of Portfolio Choice, we need to first understand how diversification works and how it can help reduce investment risk.
- In this set of notes, we will discuss:
  - how diversification can reduce risk
  - how to identify an optimal portfolio.

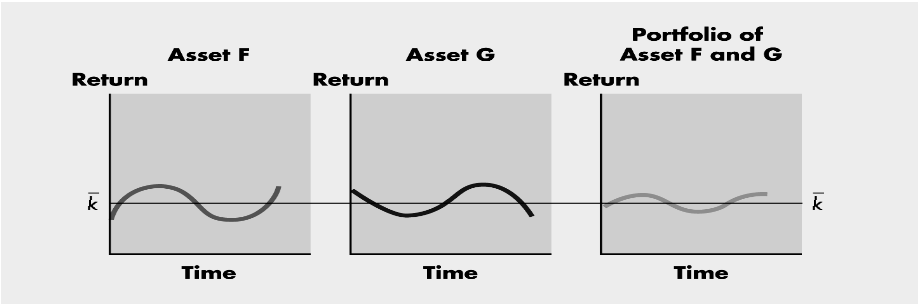
Diversification

- Instead of investing in one or two assets, investors can reduce their risk exposure by investing in a variety of assets.
- Diversification is an investment strategy designed to reduce risk exposure by combining a mix of assets which are unlikely to “move together” in the same direction.
- When economic conditions change, some assets will rise in value while others will decline in value.

Diversification

- Intuitive reason for diversification:
  - Old Wisdom: “Don’t put all your eggs in the same basket”.
- How much *overall risk* can be reduced depends on the correlation of the returns of the assets in the *portfolio*.
- We’ll be more precise about how “correlation” is measured below, but the conceptual idea is simple.





Example

- Consider two stocks from the example in the last lecture: HT and Repo.

  $(\mu_{HT}, \sigma_{HT}) = (17.4, 20.0)$

  $(\mu_{Repo}, \sigma_{Repo}) = (1.74, 13.4)$.

- Suppose we form a portfolio with \$60,000 in HT and \$40,000 in Repo.
- So the portfolio weights are w_{HT} = 0.6 and w_{Repo} = 0.4

Example

- We want to find this portfolio's
  - expected return (μ_p)
  - standard deviation of return (σ_p)
- Our objective is to show that

$$\mu_p = w_{HT}\,\mu_{HT} + w_{Repo}\,\mu_{Repo}$$

$$\sigma_p \leq w_{HT}\sigma_{HT} + w_{Repo}\,\sigma_{Repo}$$

Expected Portfolio Return μ_p

		Estimated Return		
Economy	**Prob.**	**HT %**	**Repo %**	**Port. %**
Recession	**0.10**	**-22.0**	**28.0**	**-2.00**
Below avg.	**0.20**	**-2.0**	**14.7**	**4.68**
Average	**0.40**	**20.0**	**0.0**	**12.00**
Above avg.	**0.20**	**35.0**	**-10.0**	**17.00**
Boom	**0.10**	**50.0**	**-20.0**	**22.00**

$$\begin{aligned}\mu_p = \ & 0.10\,(-0.02) + \\ & 0.20\,(\ 0.0468) + \\ & 0.40\,(\ 0.12) + \\ & 0.20\,(\ 0.17) + \\ & 0.10\,(\ 0.22) \qquad = 11.14\%.\end{aligned}$$

Portfolio Return in Different States

Here is how "Port. %" – the expected returns of a 60% HT, 40% Repo portfolio – are computed:

State	Portfolio Return (%)
Recession	0.6(-0.22) + 0.4(0.28) = -2.0%
Below Average	0.6(-0.20) + 0.4(0.147) = 4.68%
Average	0.6(0.20) + 0.4(0.0) = 12.0%
Above Average	0.6(0.35) + 0.4(-0.10) = 17.0%
Boom	0.6(0.50) + 0.4(-0.20) = 22.0%

A More Direct Method to find μ_p

- We can easily verify that the expected return of the portfolio is simply the weighted average of μ_{HT} and μ_{Repo}.
- That is

$$\begin{aligned} \mu_p &= w_{\text{HT}}\,\mu_{\text{HT}} + w_{\text{REPO}}\,\mu_{\text{REPO}} \\ &= 0.6(0.174) + 0.4(0.0174) \\ &= 11.14\% \end{aligned}$$

- Note that $\mu_{Repo} \leq \mu_p \leq \mu_{HT}$.

Standard Deviation of Portfolio Return (σ_p)

- To find the standard deviation of portfolio return, we cannot simply take the weighted average of σ_{HT} and σ_{Repo}.
- We have to compute it using the formula:

$$\sigma_P = \sqrt{\sum_{i=1}^{n} \text{Prob}_i (r_{pi} - \mu_p)^2}$$

where r_{pi} is the portfolio return (Port. %) in state i.

Standard Deviation of Portfolio Return (σ_p)

$$\sigma_p = [\,0.10\,(-0.02 - 0.1114)^2 + 0.20\,(0.0468 - 0.1114)^2 + 0.40\,(0.12 - 0.1114)^2 + 0.20\,(0.17 - 0.1114)^2 + 0.10\,(0.22 - 0.1114)^2\,]^{1/2}$$

$= 0.004457^{1/2}$

$= 0.06676$ or 6.676%

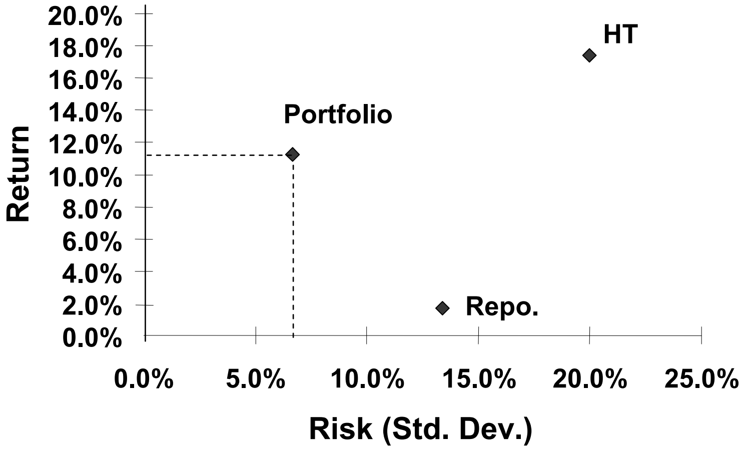
Asset Risk vs. Portfolio Risk

Portfolio	Individual Assets	
σ_p	σ_{HT}	σ_{Repo}
6.676%	20%	13.4%

Note that:

1. σ_p is lower than the weighted average of σ_{HT} and σ_{Repo}. That is $\sigma_p \leq (0.6)\sigma_{HT} + (0.4)\sigma_{Repo}$ (= 17.37%)
2. In this case, σ_p is also lower than the stand-alone risk of either stock.

(μ ,σ) of HT & Repo vs. a 60-40 Portfolio



What we learned from this example

- What have we learned so far?
  - The expected portfolio return is the weighted average of expect returns of the assets in it.

$$\mu_p = w_{HT}\,\mu_{HT} + w_{Repo}\,\mu_{Repo}$$

  - Investment risk can be reduced through diversification.

$$\sigma_p < w_{HT}\sigma_{HT} + w_{Repo}\,\sigma_{Repo}$$

- The significant risk reduction in this example is due to the fact that the returns of HT and Repo are **negatively correlated**.

How to measure "correlation"?

- Recall that the correlation coefficient between two random variables *A* and *B*
  - measures the *degree* to which their values are *linearly related*, and
  - is given by the formula $\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B}$

    where σ_{AB} is the covariance of *A* and *B.*
- Exercise: Show that ρ_{AB} = -1

** Don't confuse ρ ("rho") with the subscript "p" for "portfolio".

Diversification

- In the last example we showed you how to find μ_p and σ_p given the different states of the world (or economic conditions) and their probabilities.
- In the following example, we will work with ρ directly so that we can avoid tedious computations and focus on more important issues.
- Specifically, we want to know "What happens to μ_p and σ_p when ρ is different?"

Two-asset Portfolio

- Consider two assets *A* and *B* with (μ_A, σ_A) and (μ_B, σ_B).
- Let *p* be a portfolio consisting of these two assets, and w_A and w_B be the proportion of money invested in *A* and *B*, respectively, such that $w_A + w_B = 1$.
- Here we allow w_A and w_B to vary and we want to find the expected return and the standard deviation of return of the portfolio: μ_p and σ_p.

Formula for μ_p and σ_p

- The formulas are:

$$\mu_p = w_A\mu_A + w_B\mu_B$$

$$\sigma_p = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_A w_B\sigma_A\sigma_B}$$

$$\leq w_A\sigma_A + w_B\sigma_B$$

where ρ_{AB} is the correlation of returns between assets *A* and *B.*

Example: Portfolios with 2 assets

- Suppose the (μ,σ) of assets *A* and *B* are

$$(\mu_A,\ \sigma_A) = (0.1,\ 0.2)$$

$$(\mu_B,\ \sigma_B) = (0.16,\ 0.4)$$

- For any given ρ_{AB}, we can find all *feasible* (or attainable) combinations of *A* and *B* by varying the portfolio weights and compute the associated (μ_p,σ_p).
- We will work through three different cases: when $\rho = 1$, $\rho = 0.4$, and $\rho = -1$.

Case 1: ρ_{AB} = 1

- Let's start with the case ρ_{AB} = 1.0

$$\mu_p = w_A\mu_A + w_B\mu_B$$

$$\sigma_p = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_Aw_B\sigma_A\sigma_B}$$

$$= \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_A\sigma_B}$$

$$= \sqrt{(w_A\sigma_A + w_B\sigma_B)^2}$$

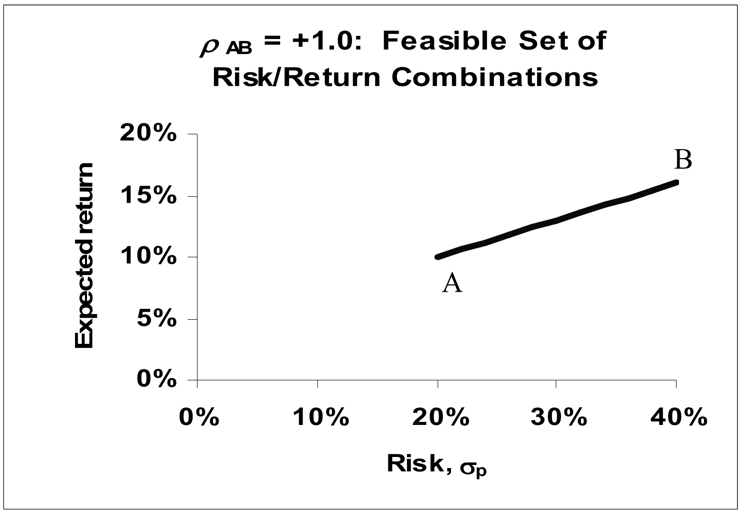
$$= w_A\sigma_A + w_B\sigma_B$$

$$\text{Note}: (a+b)^2 = a^2 + b^2 + 2ab$$

Case 1: ρ_{AB} = 1

- If ρ_{AB} = 1, there is no risk reduction effect because the returns of assets *A* and *B* are perfectly positively correlated.
- In this case, μ_p and σ_p are both linear equations: their values vary linearly with the portfolio weights on assets *A* and *B.*
- So the (μ_p, σ_p) of all feasible portfolios that can be constructed with *A* and *B* will lie on a straight line.

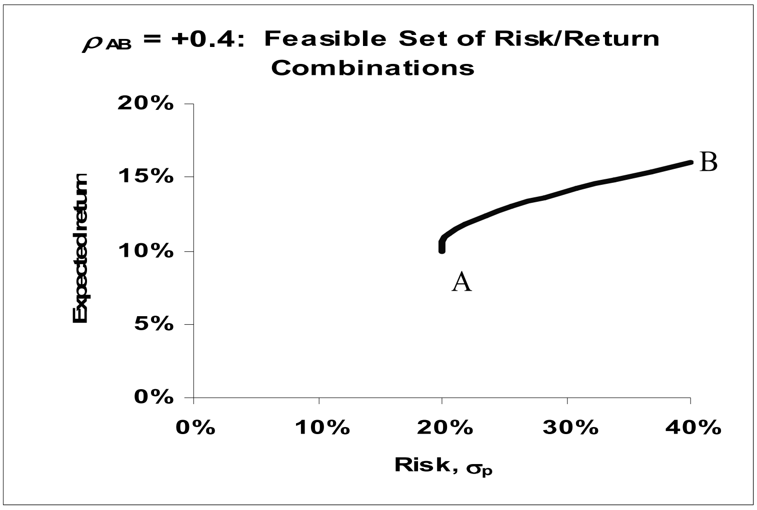
Feasible Portfolios if ρ_{AB} = 1



Case 2: ρ_{AB} = 0.4

- Suppose ρ_{AB} = 0.4
  - That is, the returns on assets *A* and *B* are positively correlated, but not perfectly so.
  - The correlation coefficient is estimated to be 0.4
- The (μ_p, σ_p) of all feasible portfolios that can be constructed with assets *A* and *B* will lie on a curve that bows to the northwest direction.

Feasible Portfolios if ρ_{AB} = 0.4



Case 3: ρ_{AB} = -1.0

- When ρ_{AB} = -1.0

$$\mu_p = w_A\mu_A + w_B\mu_B$$

$$\sigma_p = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_Aw_B\sigma_A\sigma_B}$$

$$= \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 - 2w_Aw_B\sigma_A\sigma_B}$$

$$= \sqrt{(w_A\sigma_A - w_B\sigma_B)^2}$$

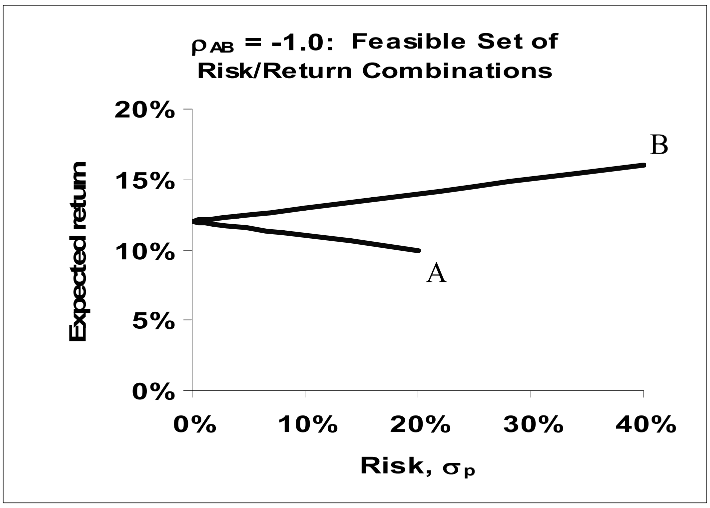
$$= | w_A\sigma_A - w_B\sigma_B |$$

$$\text{Note}: (a-b)^2 = a^2 + b^2 - 2ab$$

Case 3: ρ_{AB} = -1.0

- If ρ_{AB} = -1.0, the returns on assets *A* and *B* are perfectly negatively correlated.
- In this case, assets *A* and *B* are a perfect hedge for each other!
- Again, we want to find all feasible portfolios that can be constructed with *A* and *B* by varying the portfolio weights.
- With the appropriate portfolio weights, the investor can reduce the portfolio risk to zero (i.e. σ_p = 0)!

Feasible Portfolios if ρ_{AB} = -1

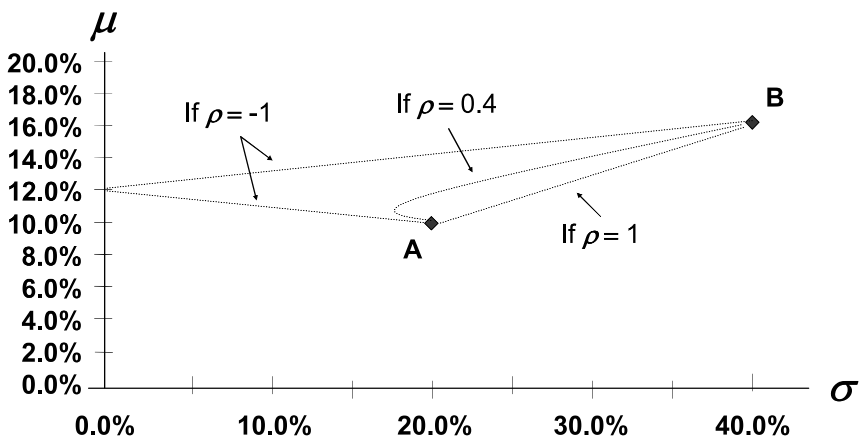


If ρ_{AB} = -1 there exists a w_A, w_B mix such that σ_p = 0

- If ρ_{AB} = -1, we can easily verify that σ_p = 0 when w_A = 2/3, w_B = 1/3.

$$\mu_p = w_A\mu_A + w_B\mu_B$$
$$= \frac{2}{3}(0.1) + \frac{1}{3}(0.16) = 0.12 = 12.0\%$$
$$\sigma_p = w_A\sigma_A - w_B\sigma_B$$
$$= \frac{2}{3}(0.2) - \frac{1}{3}(0.4) = 0$$

Summary: Feasible Set of (μ, σ)



Comparing the 3 cases

- Let's choose a combination of assets *A* and *B* so that we can compare the (μ_p, σ_p) of these 3 cases.
- Consider the portfolio weights
  $w_A = 0.3$ and $w_B = 0.7$
- The next three slides show how (μ_p, σ_p) are computed in these three cases.

Computing (μ_p, σ_p) if $\rho_{AB} = 1.0$

- If $\rho_{AB} = 1.0$

$$\mu_p = w_A \mu_A + w_B \mu_B$$
$$= 0.3(0.1) + 0.7(0.16) = 0.142 = 14.2\%$$
$$\sigma_p = w_A \sigma_A + w_B \sigma_B$$
$$= 0.3(0.2) + 0.7(0.4) = 0.34 = 34\%$$

Computing (μ_p,σ_p) if ρ_{AB} = 0.4

- If ρ_{AB} = 0.4

$$\mu_p = w_A\mu_A + w_B\mu_B$$
$$= 0.3(0.1) + 0.7(0.16) = 0.142 = 14.2\%$$
$$\sigma_p = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_Aw_B\sigma_A\sigma_B}$$
$$= \sqrt{0.3^2 0.2^2 + 0.7^2 0.4^2 + 2(0.4)(0.3)(0.7)(0.2)(0.4)}$$
$$= 0.3089 = 30.89\%$$
$$< (0.3)\sigma_A + (0.7)\sigma_B (= 34\%)$$

Computing (μ_p,σ_p) if ρ_{AB} = -1.0

- If ρ_{AB} = -1.0

$$\mu_p = w_A\mu_A + w_B\mu_B$$
$$= 0.3(0.1) + 0.7(0.16) = 0.142 = 14.2\%$$
$$\sigma_p = w_A\sigma_A - w_B\sigma_B$$
$$= 0.3(0.2) - 0.7(0.4) = 0.22 = 22\%$$

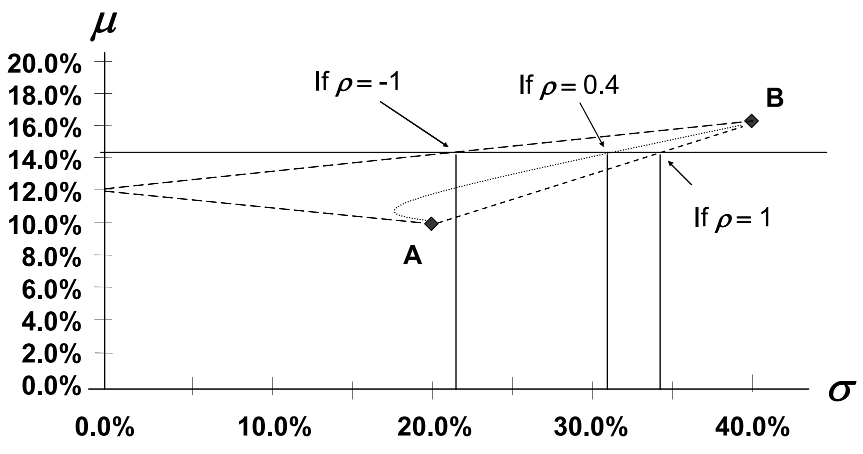
Summary of (μ_p, σ_p) if ρ_{AB} = 1.0, ρ_{AB} = 0.4 and ρ_{AB} = -1.0

- The table below summarizes (μ_p, σ_p) of the 3 cases at w_A = 0.3 and w_B = 0.7

	ρ_{AB} = 1.0	ρ_{AB} = 0.4	ρ_{AB} = -1.0
μ_p	14.2%	14.2%	14.2%
σ_p	34.0%	30.89%	22%

- Note that μ_p is the same for all cases but σ_p decreases as ρ_{AB} decreases.

Smaller ρ, more risk reduction



Summary

Diversification works because:

- The expected return of a portfolio (μ_p) is the weighted average of the expected returns of the assets in it.

$$\mu_p = w_A \mu_A + w_B \mu_B$$

- The standard deviation of a portfolio (σ_p) is generally less than the weighted average of the standard deviation of returns of the assets in it.

$$\sigma_p \leq w_A \sigma_A + w_B \sigma_B$$

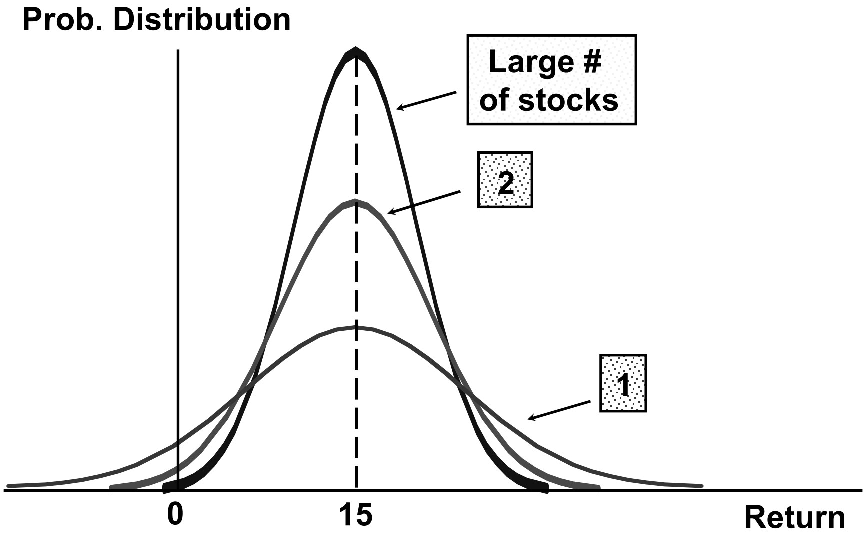
- If $\rho_{AB} = 1$, $\sigma_p = w_A\sigma_A + w_B\sigma_B$ (no risk reduction)

Summary

- By choosing the "right" assets, portfolio risk can be reduced without reducing expected return!
- How much portfolio risk can be reduced depends on the correlation of returns between the assets in the portfolio.
- The key is to combine assets with expected returns that are NOT perfectly positively correlated.
- Risk can be reduced as long as $\rho_{A,B} \neq +1.0$

μ_p and σ_p of a Large Portfolio

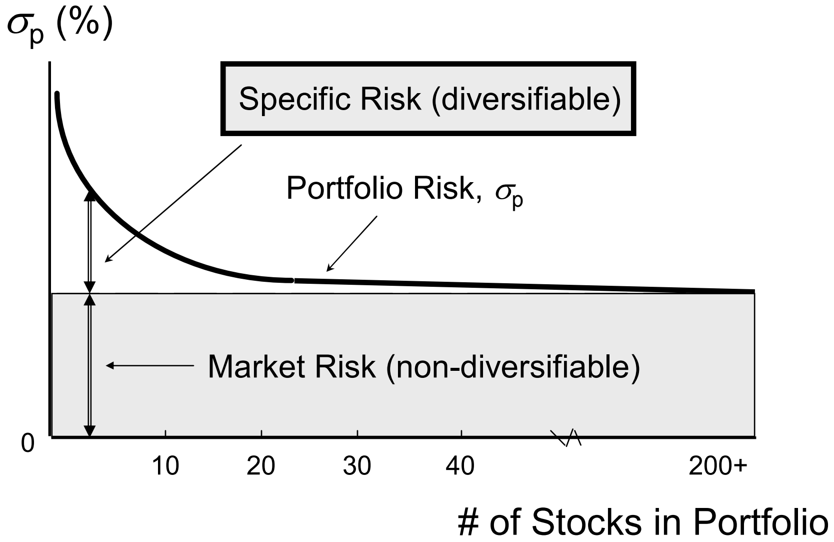
- Suppose we start with a 1-stock portfolio and keep adding more randomly selected stocks.
- μ_p would remain relatively constant since it is simply the weighted average of the expected returns of individual stocks in the portfolio.
- σ_p would decrease as long as the return of each added stock i is <u>not</u> perfectly positively correlated with that of the <u>portfolio</u> (i.e. $\rho_{ip} \neq +1.0$).



$\sigma_1 \approx 35\%,\ \sigma_{\text{Large \#}} \approx 20\%.$

Risk Reduction Effect

- Is there a limit to how much risk can be reduced through diversification? Yes.
- As more stocks are added, each additional stock has a smaller risk-reducing impact on the portfolio.
- σ_p falls very slowly after about 40 stocks are included and approaches the standard deviation of the market return.
- The lower limit of σ_p will be market risk (σ_M) – risk that cannot be diversified away.



Stand-alone risk = Market risk + Specific risk

- Market risk (or systematic risk) is the part of stand-alone risk that *cannot* be eliminated by diversification.

  Systematic risks are non-diversifiable risks.

- Specific (Unique) risk (or non-systematic risk) is the part of stand-alone risk that *can* be eliminated by diversification.

  Non-systematic risks are diversifiable risks.

Theory of Portfolio Choice

- The Theory of Portfolio Choice is developed by Harry Markowiz in the 50's.
- The key insight of Markowiz is that the relevant information about securities can be summarized by three statistical measures:
  1. The mean returns (μ)
  2. The standard deviation of returns (σ)
  3. The correlation of returns with that of other securities (ρ)

Theory of Portfolio Choice

- The Theory of Portfolio Choice is a theory of asset allocation – what assets to hold and in what proportion?
- Objective: To find an optimum portfolio, given the investor's risk preferences.
- To solve this optimization problem we need to:
  1. define the choice set
  2. model how choices are made

Theory of Portfolio Choice

- Investors care about two things: (μ,σ) of their investment portfolio.
- To model investor behavior (i.e. how investors make portfolio choices), we will
  - first define investment choices in terms of the (μ,σ) of a portfolio, and then
  - apply standard consumer theory to find the optimum portfolio of an investor given his/her risk preferences.

Portfolio with 2 Assets

- In our examples so far, we have been dealing with portfolios with two assets *A* and *B*, with

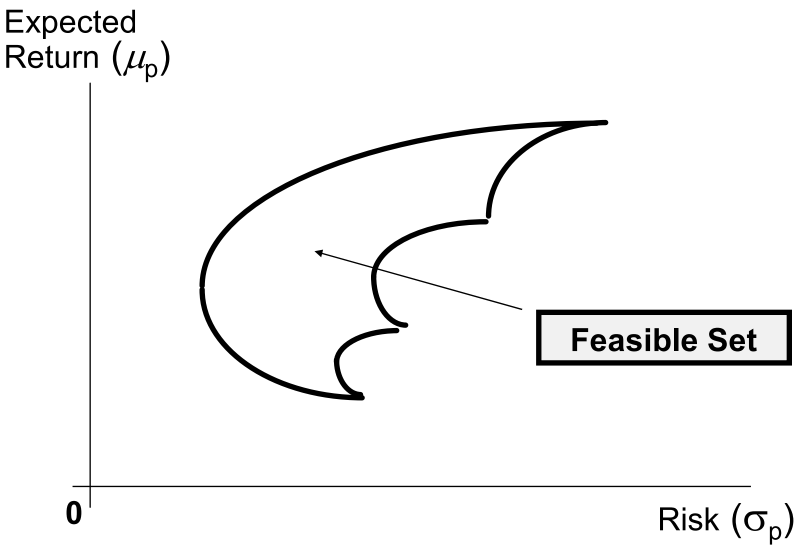
  (μ_A, σ_A) and (μ_B, σ_B)

- Given ρ_{AB}, the correlation of returns between *A* and *B,* we can construct the ***feasible set*** of (μ_p,σ_p) by varying the proportion of investment in *A* and *B.*

Portfolios with Multiple Assets

- The same idea applies to portfolios with more than two assets.
- For a given set of assets, we have to consider the correlation of returns between all possible pairs of asset and all possible portfolio weights.
- With three or more assets, the feasible set of portfolios will not be a line or a curve anymore.

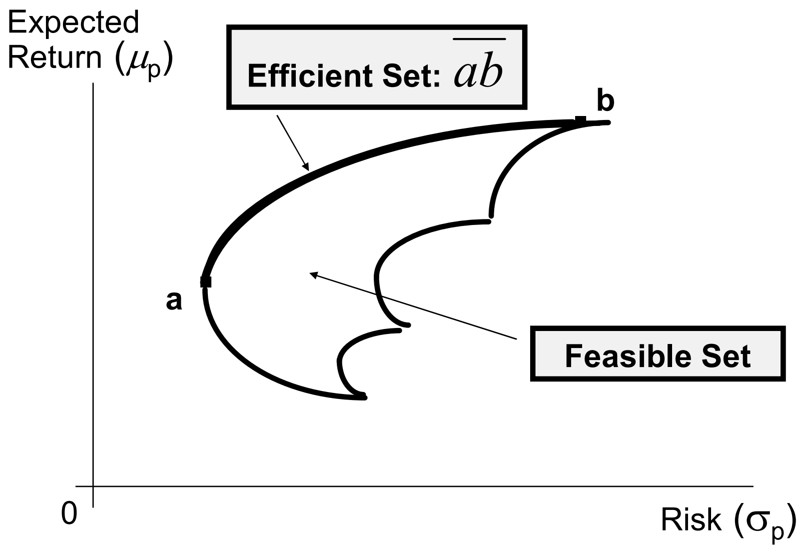
Feasible Set



Efficient Set or Frontier

- Not all feasible portfolios are efficient.
- An efficient portfolio is one that offers
  - the highest expected return (μ_p) for a given amount of risk (σ_p), or
  - the least amount of risk (σ_p) for a given expected return (μ_p).
- The set of all efficient portfolios is called the Efficient Set or Efficient Frontier (segment *ab* on the next slide).

The Efficient Frontier



Rationality and Risk-aversion

Assumptions on investor behavior

- Investors care about two things when making investment choices: risk (σ) and expected return (μ)
  - i.e. Utility function: U(σ,μ)
- Investors are rational and risk-averse in the sense that when choosing between assets or portfolios with the same expected return will always prefer the one with a lower risk.

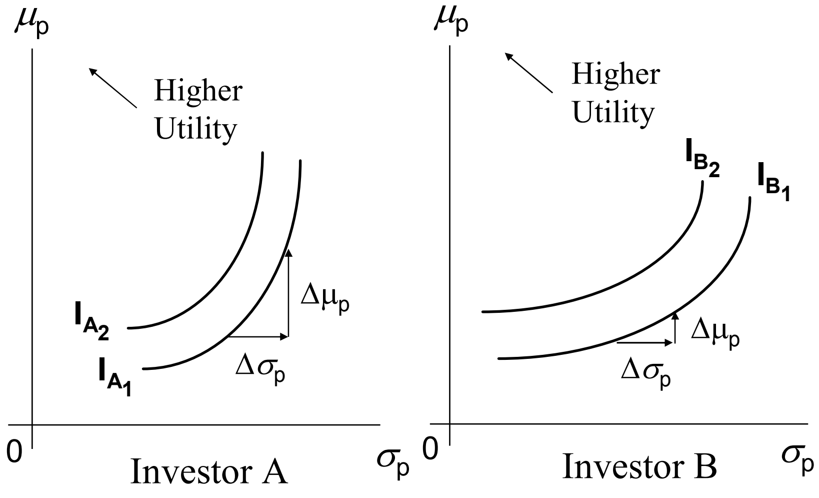
Investors as mean-variance optimizers

- In other words, investors are assumed to be mean-variance optimizer:
  - their objective is to maximize returns (μ) and minimize risk (σ)
  - i.e. they will choose a portfolio on the efficient frontier.
- Which portfolio on the frontier is optimal depends on the investor's risk preference.
- We can apply standard consumer theory to find their optimal portfolio.

Indifference Curve in μ-σ space

- Investor preferences can be represent in a μ-σ space with simple indifference curves.
- Indifference curves are upward sloping because μ is a "good" while σ is a "bad".
- The slope of an indifference curve measures the investor's willingness to trade between μ and σ – his/her attitude towards risk.
- A more risk-averse investor will have steeper indifference curves.

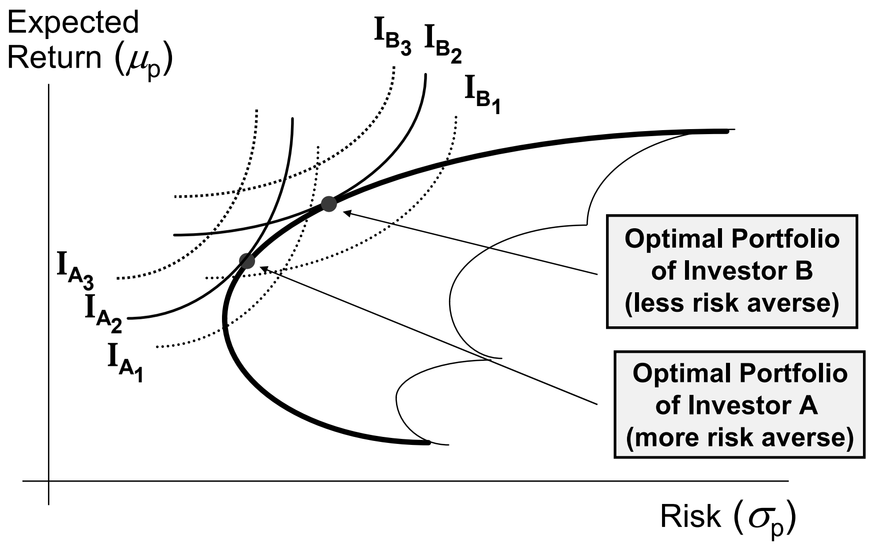
Investor A is more risk-averse than Investor B



Optimal Portfolio

- An investor's objective is to choose a portfolio on the efficient frontier that can put him/her on the highest indifference curve.
- An investor's optimal portfolio is one at which the efficient frontier is tangent to the highest indifference curve.
- In our example, Investor A is more risk-averse and the (μ,σ) of her optimal portfolio is lower than that of Investor B.
- i.e., a more risk-averse investor will choose a portfolio with lower risk and lower return.

Optimal Portfolio



Summary

- The key to diversification is the correlation of returns across assets.
- The "relevant" risk of an individual asset i is <u>not</u> σ_i (its stand-alone risk).
- Why? Because the "Specific Risk" of an asset is diversifiable – that is, it can be diversified away through a portfolio.
- How much portfolio risk can you reduce by adding one more asset depends on *the* correlation between *the return of that asset* and *the return of the portfolio*.

Summary

Defining and measuring risk

- Risk can be decomposed into two parts:
  - Risk that can be diversified away by investors
  - Risk that cannot be diversified away.
- Risk does not “add up”.
  - By combining a careful selection of assets, different risks neutralize each other.
  - Risk reduction is more effective if correlation of returns between assets is small.

Next Lecture

- One central question in finance is:

  “*How do we measure the non-diversifiable risk of an investment?”*
- Why measure non-diversifiable risk? Because the “appropriate” return of an investment is determined by its risk premium, and there should only be a premium on risk that is not diversifiable.
- The most widely used model in finance is the Capital Asset Pricing Model (CAPM).

Exercise

- Show that
  1. The covariance of the returns on HT and Repo is

     $\sigma_{HT,Repo}$ = -0.0268
  2. $\rho_{HT,Repo}$ = -1
  3. σ_p = 0 if w_{HT} = 0.4 and w_{Repo} = 0.6

Expected Portfolio Return μ_p

		Estimated Return		
Economy	**Prob.**	**HT %**	**Repo %**	**Port. %**
Recession	**0.10**	**-22.0**	**28.0**	**0.08**
Below avg.	**0.20**	**-2.0**	**14.7**	**0.08**
Average	**0.40**	**20.0**	**0.0**	**0.08**
Above avg.	**0.20**	**35.0**	**-10.0**	**0.08**
Boom	**0.10**	**50.0**	**-20.0**	**0.08**
mean =		**0.174**	**0.0174**	**0.08**
variance =		**0.0401**	**0.0179**	**0.00**
s.d. =		**0.2004**	**0.1336**	**0.00**

Exercise

1. Covar(HT, Repo) =

   0.1 (-0.22-0.174)(0.28-0.0174) +

   0.2 (-0.02-0.174)(0.147-0.0174) +

   0.4 (-0.20-0.174)(0.0-0.0174) +

   0.2 (-0.35-0.174)(-0.10-0.0174) +

   0.1 (-0.50-0.174)(-0.20-0.0174)

   = -0.0268

Exercise

2. $\rho_{HT,Repo} = \sigma_{HT,Repo} / \sigma_{HT}\sigma_{Repo}$

   $= -0.0268 / (0.2004)(0.1336)$

   $= -1$

3. If $w_{HT} = 0.4$ and $w_{Repo} = 0.6$

   $\sigma_p = |\ (0.4)\ \sigma_{HT} - (0.6)\ \sigma_{Repo}\ |$

   $= |\ (0.4)(0.2004) - (0.6)(0.1336)\ |$

   $= 0$

Feasible (μ ,σ) of HT-Repo Portfolio