Lecture 2
Time Value of Money

- Time lines (timing of cash flow)
- Future Value (Compounding)
- Present Value (Discounting)
- Annuities (FV & PV of Annuities)
- Amortization (Installment) Loans
- Effective Annual Rates (EAR)

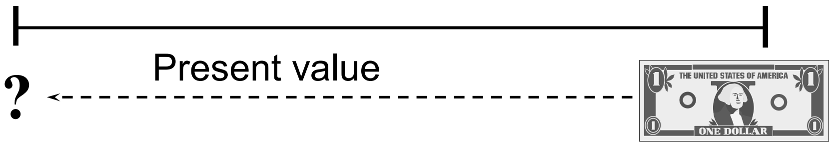
Time Value of Money

- The Time Value of Money is one of the fundamental concepts of finance.
- $1 received today is worth more than $1 received in the future because of
  - Opportunity Costs
  - Time Preference

To measure the Time Value of Money, we can:

- Translate $1 today into its equivalent in the future (Compounding).

- Translate $1 in the future into its equivalent today (Discounting).



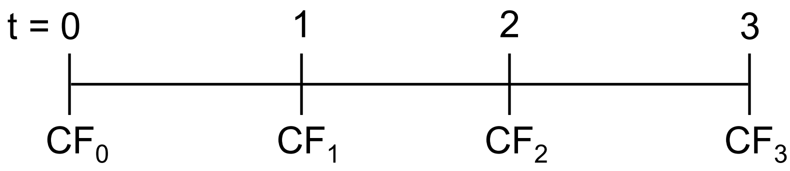
Future Value & Present Value

- Future Value: how much a sum of money is worth at a specific time in the future, assuming a given rate of interest?
- Present Value: how much a future payment (or a series of future payments) is worth today, assuming a given rate of interest?

  The payment(s) is *discounted* to reflect the time value of money and other relevant factors such as risk.

Time Line of Cash Flows

To visualize the timing of cash flow associated with a situation, we can draw a time line.

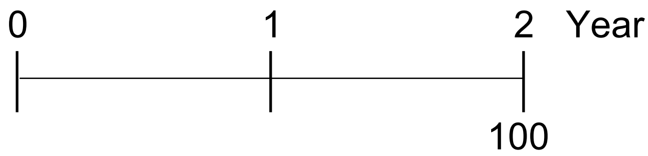


Tick marks: Time 0 (t=0) is today or current period; Time 1 (t=1) is the end of Period 1 or the beginning of Period 2.

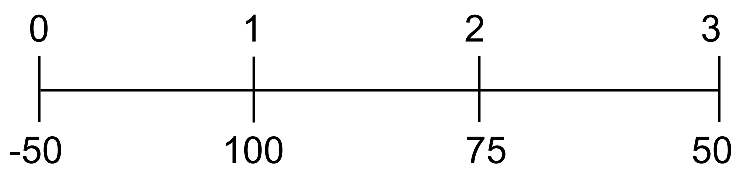
CF_t = cash flow at time t

Examples of Time line

- A lump sum of $100 to be paid in 2 years



- An uneven CF of -$50 at t = 0 and $100, $75, and $50 at the end of years 1 – 3.



Ways to Find FV & PV

- Time Line Solution

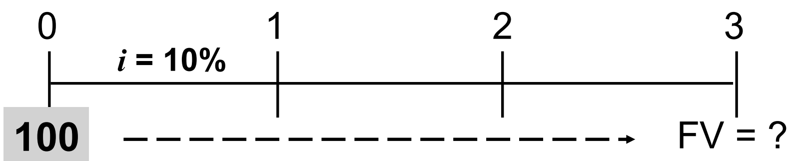
  Solve the cash flows of the time line
- Numerical Solution

  Solve using the appropriate equation / formula
- Financial Calculators w/ built-in functions
- Spreadsheet Solution

  Using spreadsheet functions
- Financial Tables (obsolete)

What's the FV of \$100 after 3 years if i = 10%?

Suppose you deposit \$100 in an account that earns 10% interest, assume annual compounding. How much will you have in the a/c after 3 years?



Compounding is the process of earning interest on the principal and the interest earned.

The Idea of Compounding

After 1 year:

$$FV_1 = PV + INT_1$$
$$= PV + PV \times i$$
$$= PV \times (1 + i)$$
$$= \$100 \times (1.10) = \$110.00.$$

After 2 years:

$$FV_2 = FV_1 \times (1 + i)$$
$$= PV \times (1 + i)^2$$
$$= \$100 \times (1.10)^2 = \$121.00.$$

After 3 years:

$$FV_3 = FV_2 \times (1 + i)$$
$$= PV \times (1 + i)^3$$
$$= \$100 \times (1.10)^3 = \$133.10.$$

Time Line Solution

0 — 10% — 1 — 2 — 3

End of yr. amount

100 ×1.10 ×1.10 ×1.10

= 110.00 = 121.00 **= 133.1**

$$FV_3 = 100 \times (1.10 \times 1.10 \times 1.10)$$
$$= 100 \times (1.10)^3$$
$$= 133.1$$

Using Formula

- The formula for computing FV is:

$$FV_n = PV \times (1 + i)^n$$

where FV_n is the FV at the end of period n.

- In our example:
  - PV = $100
  - i = 0.1
  - n = 3

FV = $100 $(1+ 0.1)^3$

= 100 x 1.331 = 133.1

Financial Calculator (HP-10B) to find FV:

INPUTS	3	10	-100	0	
	N	**i/YR**	**PV**	**PMT**	**FV**
OUTPUT					**133.10**

•Press “Shift” then “C” (clear all) to clear memory (sets everything to 0) before entering data.
•Either PV or FV must be negative. Here, “put in” $100 and “take out” $300 three years later.

For tutorial, see http://www.tvmcalcs.com/calculator_index

Nominal Rate: i_{Nom}

- Interest rates stated in contracts or quoted by banks or brokers are typically nominal rates.
  - Nominal Rates (i_{Nom}) are **annual rates**.
- The example above assumes annual compounding, i.e. interest is computed once a year.
- If interest is computed more than once a year, we need to use the periodic rate which is based on the # of periods per year.

Periodic Rate: i_{Per}

- Periodic Rate is simply

$$i_{Per} = i_{Nom}/m,$$

where m is number of periods per year.

- m = 4 for quarterly, = 12 for monthly, and = 360 or 365 for daily compounding.
- Example: Suppose the finance charge of your credit card has a nominal rate of 24% APR. Since interest (finance charge) is computed monthly, the periodic rate is 24%/12 = 2% per month.

FV Formula For Compounding

- General Formula:

  $FV = P \times (1+i/m)^{mn}$

  where P = principal (or PV), i = annual rate of interest, n = # of years, m = # of periods per year.

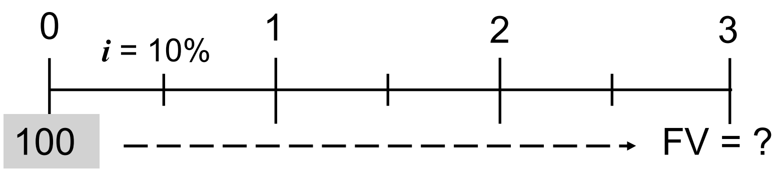
- Annual Compounding (m = 1):

  $FV = P \times (1+i)^{n}$

- Semi-annual Compounding (m = 2):

  $FV = P \times (1+i/2)^{2n}$

What is the FV of \$100 after 3 years if i = 10%, semi-annual compounding?



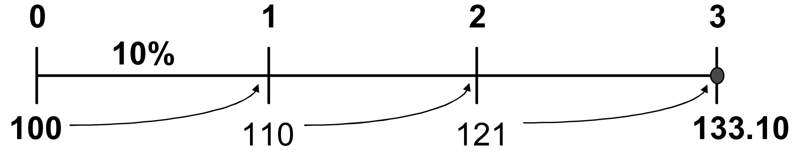
- Total number of periods: 3x2 = 6.
- Since interest is earned every 6 months, we need to use the semi-annual rate 10%/2 = 5%

FV with Semi-annual Compounding

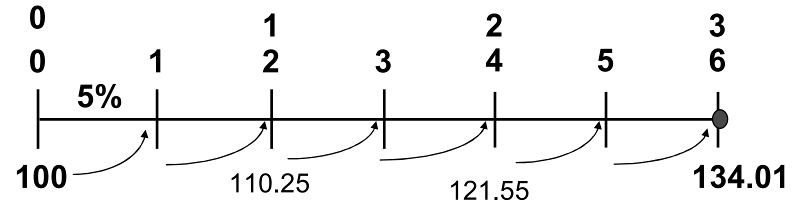
In this case, $P = 100$, $i = 10\%$, $m = 2$, $n = 3$

$$FV = P \times (1 + i/m)^{mn}$$
$$= 100 \times (1 + 0.1/2)^{2x3}$$
$$= 100 \times 1.05^{6}$$
$$= 100 \times 1.34009$$
$$= 134.01$$

Annually: $FV_3 = {}^{\$}100(1.10)^3 = \$133.10$.



Semi-annually: $FV_6 = {}^{\$}100(1.05)^6 = \$134.01$.



Annual Compounding

If you deposit $100 in an account earning 6%, how much would you have in the account after 5 years?

PV = 100 (m=1, n=5) FV = 133.82

0 5

$$FV = PV\,(1 + i\,)^n$$

$$FV = 100\,(1.06)^5 = 133.82$$

Quarterly Compounding

If you deposit $100 in an account earning 6% with *quarterly compounding*, how much would you have in the account after 5 years?

PV = 100 (m=4, n=5) FV = 134.68

0 5

$$FV = PV\,(1 + 0.06/4)^{4x5}$$

$$FV = 100\,(1.015)^{20} = 134.68$$

Monthly Compounding

If you deposit $100 in an account earning 6% with *monthly compounding*, how much would you have in the account after 5 years?

PV = 100 (m=12, n=5) FV = 134.89

0 ——————————————— 5

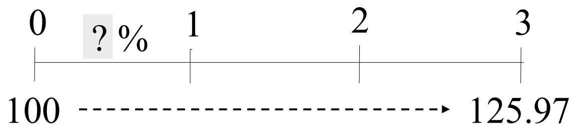
$$FV = PV\,(1 + 0.06/12)^{12x5}$$
$$FV = 100\,(1.005)^{60} = 134.89$$

Conclusion

PV = 100 APR=6%	Annual (m=1)	Qtrly (m=4)	Mthly (m=12)
FV_5 =	$100(1.06)^5$ = $133.82	$100(1.015)^{20}$ = $134.68	$100(1.005)^{60}$ = $134.89

- The FV of a lump sum will be larger if interest is compounded more frequently because interest is earned on interest more often.

Finding the interest rate needed for a lump sum to grow to some FV in *n* years



Assume annual compounding

$$FV_n = PV \times (1+ i)^n$$

$$125.97 = 100 \times (1+ i)^3$$

Find i.

What interest rate would be needed for $100 to grow to $125.97 in 3 years?

Solution:

$$\$100(1 + i)^3 = \$125.97$$
$$(1 + i)^3 = \$1.2597$$
$$1 + i = (1.2597)^{1/3}$$
$$1 + i = 1.08$$
$$i = 8\%.$$

Note:

$$\text{If } x^k = y, \text{ then } x = \sqrt[k]{y} = y^{1/k}$$

What if interest is compounded semi-annually?

$^{\$}100(1 + i/2)^{3x2} = {}^{\$}125.97$

$(1 + i/2)^6 = {}^{\$}1.2597$

$1 + i/2 = (1.2597)^{1/6}$

$1 + i/2 = 1.039229$

$i = 2 \times 0.039229$

$i = 7.845\%$ ← this is an annual rate

If interest is compounded more frequently, the interest rate needed for $100 to grow to $125.97 in 3 years will be lower.

Caution: mistake to avoid

If you set up your equation this way:

$^{\$}100(1 + i)^{3x2} = {}^{\$}125.97$

$(1 + i)^6 = {}^{\$}1.2597$

$1 + i = (1.2597)^{1/6}$

$1 + i = 1.039229$

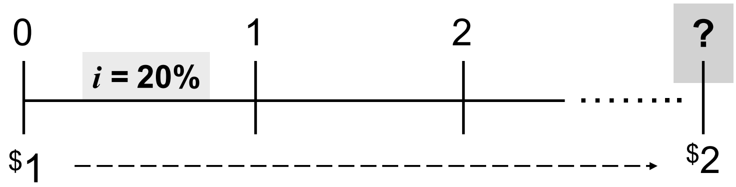
$i = 0.039229$

The result is a semi-annual rate and you have to multiply it by 2 to get the annual rate:

$2 \times 0.039229 = 7.845\%$

Finding the “Time to Double”

- How long will it take a fixed amount to double at some interest rate, compounded annually?



$$FV = PV(1 + i)^n$$
$$\$2 = \$1(1 + 0.20)^n$$
$$2 = (1.2)^n$$

We want to find *n.*

Finding the “Time to Double”

- To solve for *n,* recall that

If $y = x^a$ then $log\ y = log\ x^a$

$$= a \cdot log\ x$$

- So we take the *log* of both sides

$$(1.2)^n = 2$$
$$n \cdot log(1.2) = log(2)$$
$$n = log(2) / log(1.2)$$
$$n = 0.301 / 0.0792$$
$$= 3.8 \text{ years.}$$

Financial Calculator

INPUTS		20	-1	0	2
	N	i/YR	PV	PMT	FV
OUTPUT	3.8				

Estimating "time to double": The Rule of 72

- We can also get a rough estimate of the number of years it takes a sum of money to double at a given rate of interest using the "Rule of 72" .
- Rule of 72: The number of years it take an initial sum of money to double at i% interest rate is roughly equal to $72/i$.
- E.g. At 12% interest, it will take approximately 72/12 = 6 years for an initial sum of money to double.

A More Complicated Example

- Suppose you placed \$100 in an account that pays 9.6% interest, compounded monthly. How long will it take for your account to grow to \$500?

$FV = PV\,(1 + i/12)^k$ where k = # of months

$500 = 100\,(1+ 0.008)^k$ (Note: 9.6% / 12 = 0.008)

$5 = (1.008)^k$

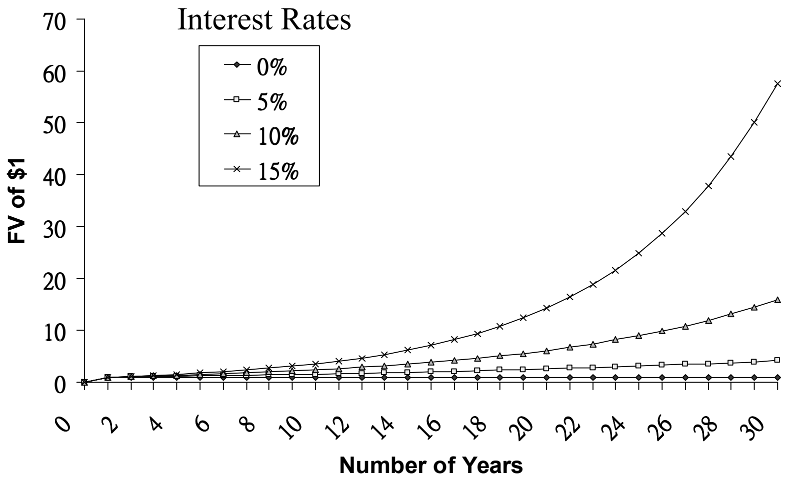
$\log(5) = \log(1.008)^k$

$\log(5) = k \cdot \log(1.008)$

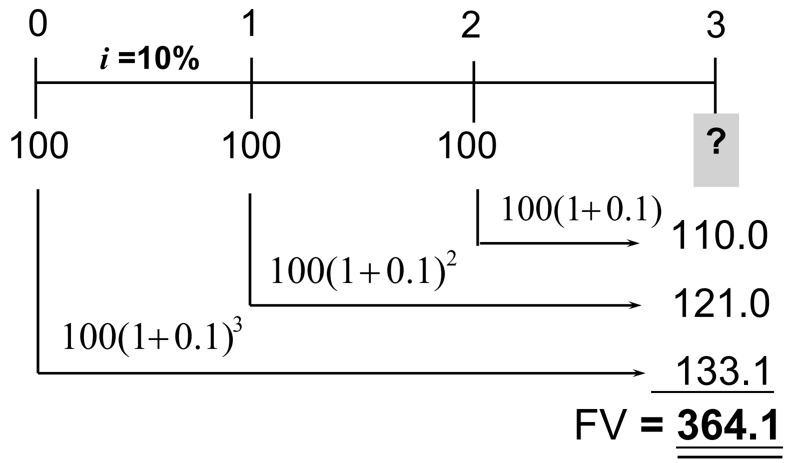
$0.6990 = 0.003461 \cdot k$

k = 202 months (16 yrs & 10 mths)

Future Values @ different *i*



FV of a Series of Cash Flow



PV and Discounting

- The discounted present value of a dollar to be received in ***n*** years measures how much that dollar is worth **today**.
- The concept of PV provides a common unit of measurement so that the value of different financial instruments that pay different amounts at different times can be compared.

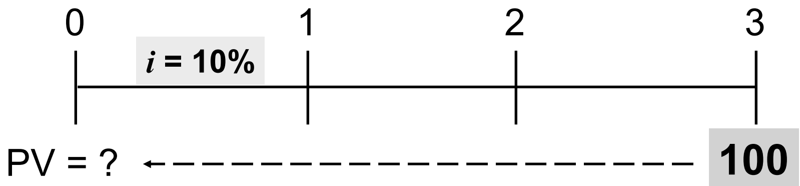
Present Value Formula

- To find PV we use *discounting*, which is the reverse of compounding.
- Rearrange the FV formula for PV

$FV_n = PV(1 + i)^n$

$$PV = \frac{FV_n}{(1+i)^n} = FV_n\left(\frac{1}{1+i}\right)^n$$

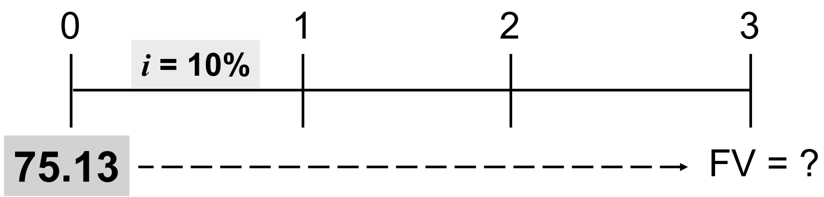
Example: What is the PV of $100 due in 3 years if *i* = 10%?



$$PV = \frac{100}{(1+0.1)^3}$$

$$= 100\ (0.7513) = 75.13$$

Note: You can check your answer by doing the reverse



Check:

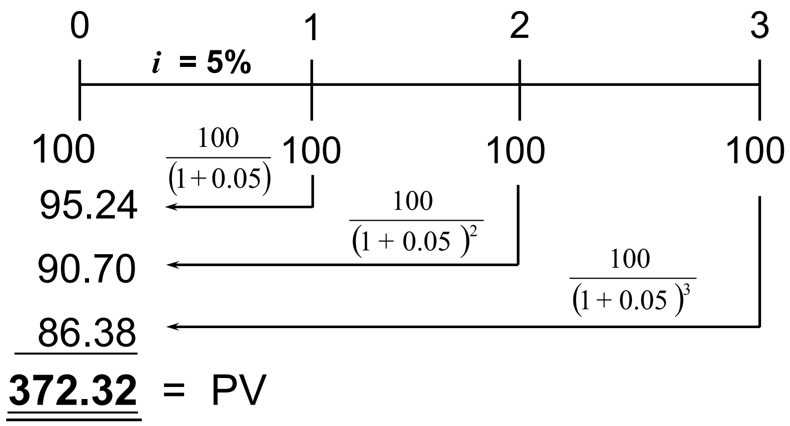
$$\begin{aligned} FV &= 75.13(1+0.1)^3 \\ &= 75.13(1.331) = 100 \end{aligned}$$

Financial Calculator Solution

INPUTS	3	10		0	100
	N	i/YR	PV	PMT	FV
OUTPUT			-75.13		

Either PV or FV must be negative.
Here PV = -75.13. Put in $75.13 today, take out $100 after 3 years.

PV of a series of cash flows



Exercises

1. Suppose i = 8%. Are you better off getting
   - a). $100 in one year, or
   - b). $110 in two years?
2. Suppose you win a lottery that pays $100,000 now and $100,000 a year in the next 3 years. If someone offers to buy your winning ticket for $360,000, should you take the offer if the current interest rate is 5%?

Exercise #1

- To compare these 2 options, we need to compute their PV.
- At i = 8%

a). $100 in one year

PV = 100 / (1.08) = __________

b). $110 in two years

PV = $110 / (1.08)^2$ = __________

Ans: Option ___ is better.

Exercise #2

- In this case, we want to compare

a). A lump sum of $360,000 today with

b). The PV of the lottery payments which is a series of cash flow

- The PV of the lottery payments discounted at 5% is

PV = $100,000 + $100,000 / 1.05 + $100,000 / 1.05^2 + $100,000 / 1.05^3

= $372,323.76

- Ans: You should reject the offer.

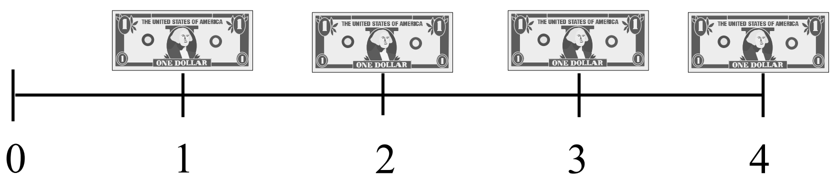
Annuities

- An annuity is a series of equal cash flow (payments or receipts) occurring over a specified number of periods.
- There are two basic types of annuities:
  - For an Ordinary (or Deferred) Annuity, the payments or receipts occur at the end of each period.
  - For an Annuity Due, the payments or receipts occur at the beginning of each period.

Ordinary Annuity

- An Ordinary (or deferred) Annuity: a series of equal cash flow (payments or receipts) occurring at the end of each period.

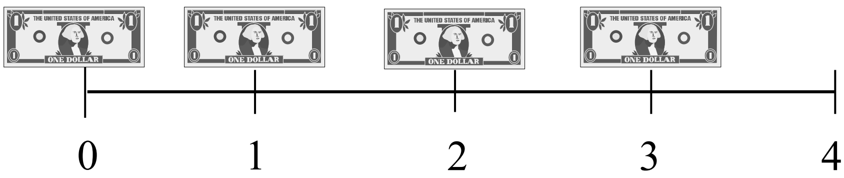
  (e.g. home loans, car loans, saving plans)



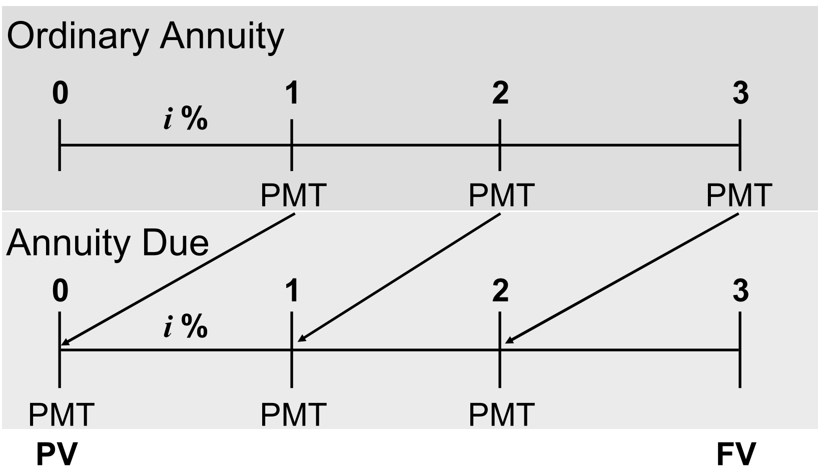
Annuity Due

- An Annuity Due: a series of equal cash flow (payments or receipts) occurring at the beginning of each period.

  (e.g. rental payments, insurance premium)



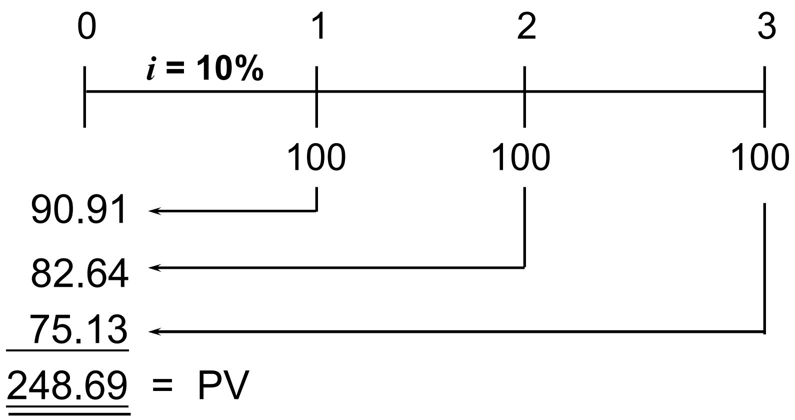
What is the difference between an ordinary annuity and an annuity due?



Examples of Annuities

- If you borrow money to buy a house or a car, you pay a stream of **equal** payments until the loan is paid off.
- If you buy a bond, you will receive equal interest (coupon) payments over the life of the bond.
- If you open an annuity account to save for retirement or college funds for your children, you typically pay-in an equal amount monthly for a specific number of years.

PV of An Annuity



Annuity Formula

- With only a small number of payments, we can compute the present value of an annuity by simply adding up the PV of each payment.
- For annuities with a large number (say, 20 or more) payments, the computation can become very laborious.
- Fortunately, there is a formula we can use.

Annuity Formula: PV

$$PV_A = PMT\left(\frac{1}{i} - \frac{1}{i(1+i)^n}\right)$$

- The expression in the brackets is called an annuity factor: *the present value of a n-year ordinary annuity of $1 a year.*

PV of Annuity (By Formula)

What is the PV of $100 at the end of each of the next 3 years, if the opportunity cost is 10%?

$$PV_A = PMT\left(\frac{1}{i} - \frac{1}{i(1+i)^n}\right)$$

$$= 100\left(\frac{1}{0.1} - \frac{1}{0.1(1+0.1)^3}\right) = 248.69$$

Amortization

- Amortization is a systematic payment plan so that a loan is paid off over a specific period.
- If a loan is to be repaid in equal periodic amounts, it is called an amortized loan. e.g. car loans, mortgages
- So an amortized loan IS an annuity.
- An amortized loan is structured in such a way that the discounted PV of the equal periodic (or installment) payments will add up to the principal of the loan.

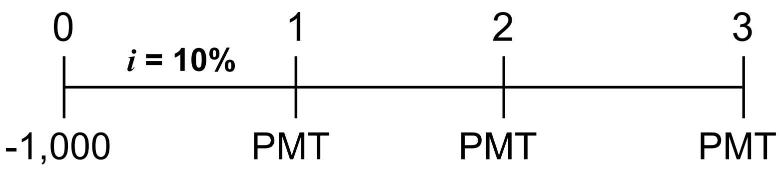
Amortization

- Example: Consider a $1,000 loan at 10% annual rate with 3 equal annual payments.

- To construct the amortization schedule of this loan, we need to find its installment payments.

$$1{,}000 = \frac{\text{PMT}}{1+0.1} + \frac{\text{PMT}}{(1+0.1)^2} + \frac{\text{PMT}}{(1+0.1)^3}$$

Find the required payments



	N	i/YR	PV	PMT	FV
INPUTS	3	10	-1,000		0
OUTPUT				402.11	

Mathematical Solution:

Use PV of Annuity formula.

Given i = 10%, n = 3

$$PV_A = PMT\left(\frac{1}{i} - \frac{1}{i(1+i)^n}\right)$$

$$1{,}000 = PMT\left(\frac{1}{0.1} - \frac{1}{0.1(1+0.1)^3}\right)$$

$$PMT = 402.11$$

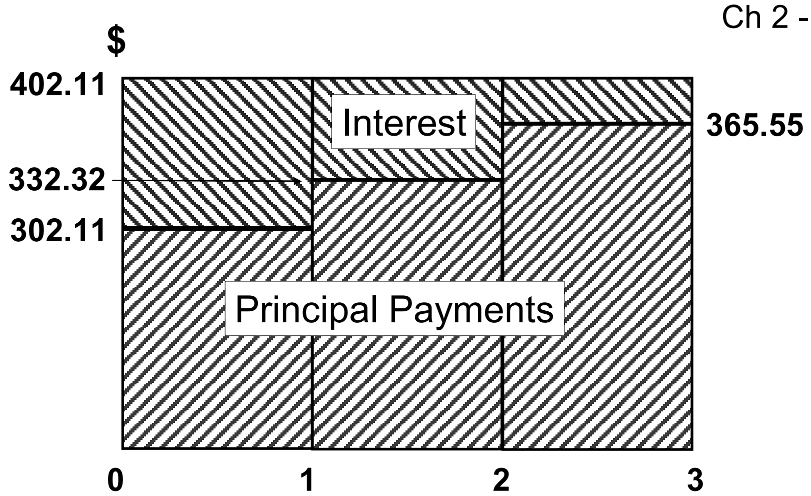
YR	BEG BAL	ANNUAL PMT	INT PMT	PRIN PMT	END BAL
1	$1,000	$402	$100	$302	$698
2	698	402	70	332	366
3	366	402	37	366	0
TOT		1,206.34	206.34	1,000	

Check:

$$PV = \frac{402.11}{1.1} + \frac{402.11}{(1.1)^2} + \frac{402.11}{(1.1)^3}$$

$$= 365.55 + 332.32 + 302.11$$

$$= 999.98$$



Level payments. Interest payments decline at an increasing rate because outstanding balance is declining.

Mortgage Payment Example

- If you borrow $1,000,000 at 3% fixed interest for 20 years to buy a house, what will be your monthly mortgage payment?
- In this case, we need to use the monthly (periodic) rate = 3%/12 = 0.0025 and let n = 20x12 = 240.

Mortgage Payment Example

Mathematical Solution:

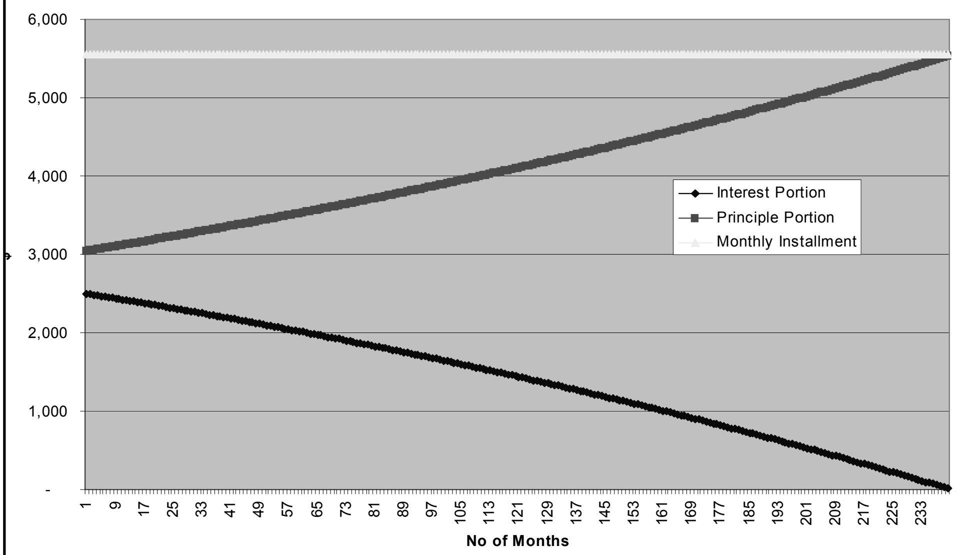
Given $\boldsymbol{i}$ = 0.03/12 = 0.0025, n = 20x12 = 240, and PV = 1,000,000. Find PMT.

$$PV_A = \frac{PMT}{i}\left(1 - \frac{1}{(1+i)^n}\right)$$

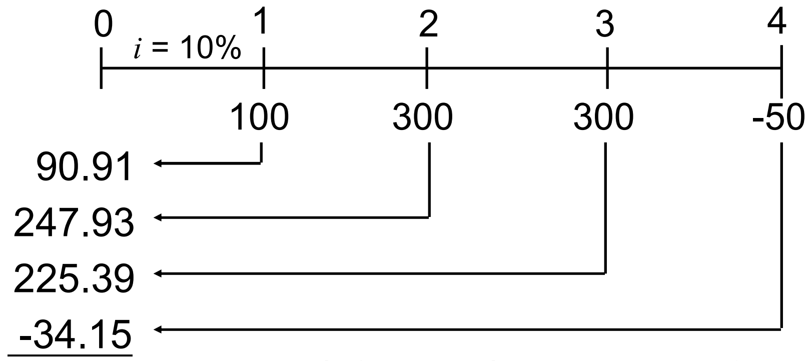
$$1{,}000{,}000 = \frac{PMT}{0.0025}\left(1 - \frac{1}{(1+0.0025)^{240}}\right)$$

$$PMT = 5{,}545.98$$

PV =	1,000,000
Annual Interest =	3%
No. of Years =	20
PV = PMT { [1 - 1 / (1+i)^n] / i }	
Monthly PMT =	5,545.98



What is the PV of this <u>uneven</u> cash flow stream if *i* = 10%?

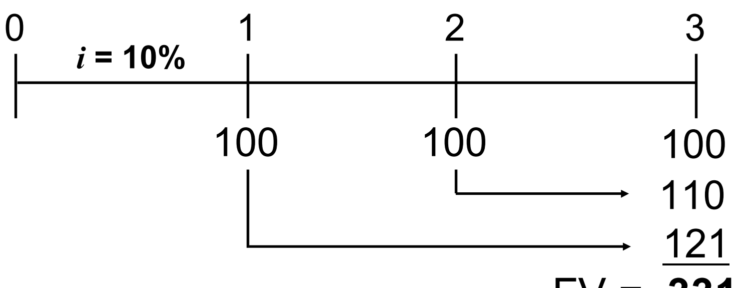


530.08 = PV

* Cannot * use annuity formula for uneven cash flow.

FV of Annuities

What's the FV of a 3-year ordinary annuity of $100 at 10%?



FV = **331**

Financial Calculator Solution

INPUTS	3	10	0	-100	
	N	i/YR	PV	PMT	FV
OUTPUT					331.00

Have payments but no lump sum PV, so enter 0 for present value.

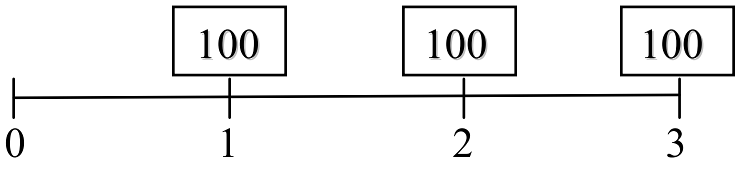
Annuity Formula: FV

- Again, if the number of payments is large the computation can get very laborious.
- There is a formula for the FV of Annuity we can use:

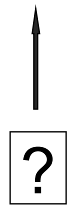
$$FV_A = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

Example: FV of Annuity

Suppose you invest $100 at the end of the next 3 years. How much will you have after 3 years if interest rate is 10%?



$$FV_A = 100\left(\frac{(1+0.1)^3 - 1}{0.1}\right) = 331$$

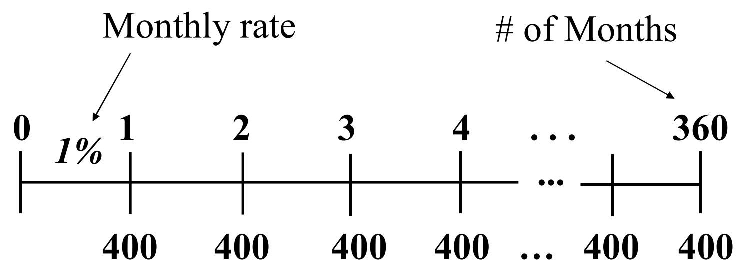


Example: Saving for Retirement

- Suppose you invest $400 at <u>the end</u> of each month for the next 30 years at 12%. How much will you have **at the end** of year 30?
- We want to find the FV of an ordinary annuity.
- There will be 30x12=360 monthly payments, and the monthly interest rate (periodic rate) is 12%/12=1%.
- Using the FV formula for annuity, we let i = 1% and n = 360.

Retirement Example

- Time line of the retirement plan:



Retirement Example

Mathematical Solution:

i =1%, n = 360

$$FV_A = PMT\left(\frac{(1+i)^n - 1}{i}\right)$$

$$= 400\left(\frac{(1+0.01)^{360} - 1}{0.01}\right)$$

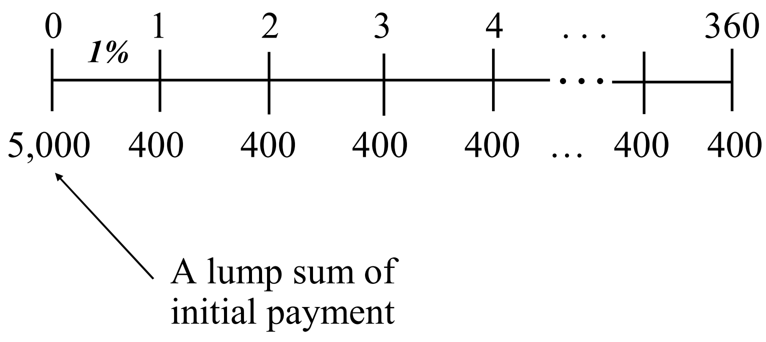
$$= 1{,}397{,}985.65$$

Retirement Example Revisited

- For retirement savings, one typically opens an account with an *initial payment* and then put in regular payments over a number of years.
- Consider the same retirement example above. Suppose you put in an **initial lump sum** of \$5,000 (at t = 0) in addition to the 360 subsequent monthly payments.
- How much will you have in your account after 30 years (at t = 360)?

Retirement Example Revisited

- The time line of a retirement plan with an initial lump sum payment:



Retirement Example Revisited

- The amount of money in your retirement account after 30 years will be

$$FV_A = 5{,}000(1+0.01)^{360} + 400\left(\frac{(1+0.01)^{360} - 1}{0.01}\right)$$

$$= 179{,}748.2 + 1{,}397{,}985.65$$

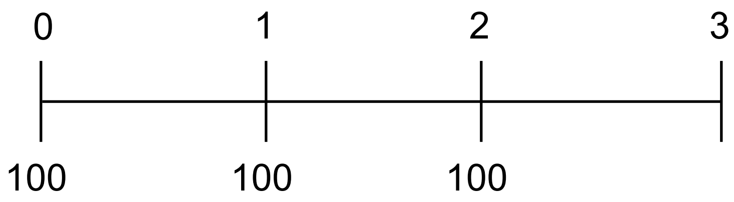
$$= 1{,}415{,}733.85$$

Retirement Example Revisited

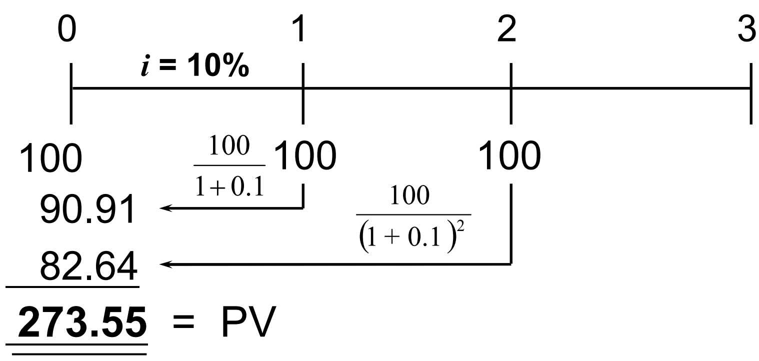
- What if the initial payment is also \$400, the same amount as the subsequent monthly payments?
- That mean there will be 361 equal payments of \$400 with the first one starting at t = 0.
- This will make the saving plan a simple <u>annuity due</u> (sort of).

Annuity Due

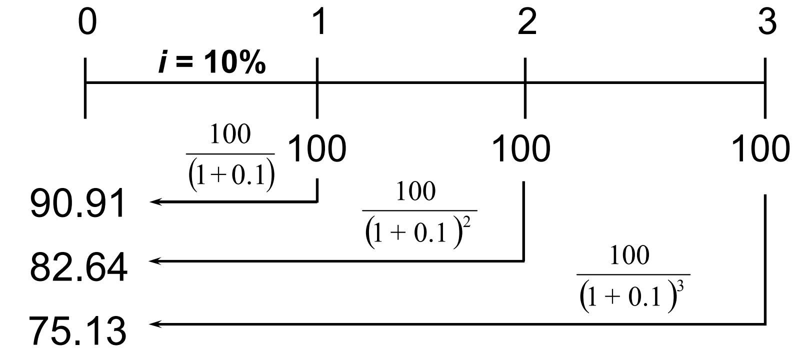
An annuity due is a series of **equal** cash flow occurring at the **beginning** of each period.



What is the PV of a 3-year annuity due of $100 at 10%?



If this was an 3-year ordinary annuity of $100, what would its PV be?



248.69 = PV

Notice that \$248.69 x (1+10%) = 273.55, which is the answer in the previous slide!

PV: Ordinary Annuity vs. Annuity Due

- If this was an ordinary annuity, its PV would be:

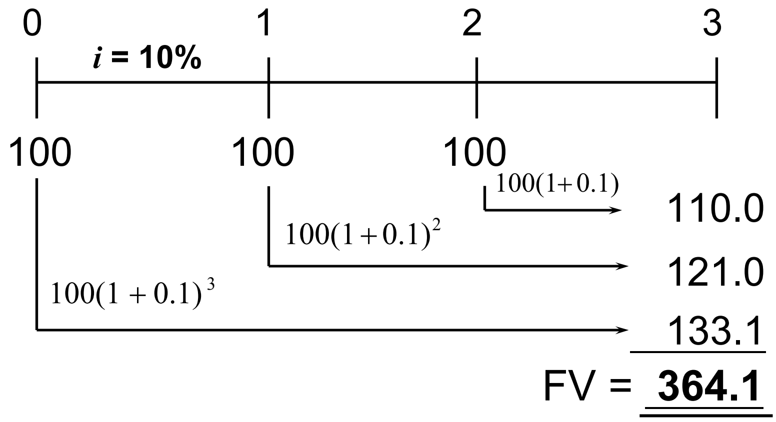
$$PV_A = \frac{100}{1+0.1} + \frac{100}{(1+0.1)^2} + \frac{100}{(1+0.1)^3}$$

- If we multiple PV_A by 1+ *i,* we have

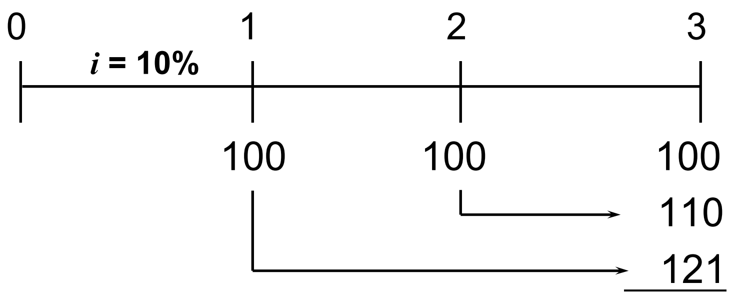
$$PV_A(1+i) = 100 + \frac{100}{1+0.1} + \frac{100}{(1+0.1)^2}$$

- That is, all payments would occur one period sooner → the PV of an annuity due!

What is the FV of a 3-year annuity due of $100 at 10%?



If this was a 3-year ordinary annuity of $100, what would its FV be?



FV = **331**

Notice that $331 x (1+10%) = 364.1, which is the answer in the previous slide!

FV: Ordinary Annuity vs. Annuity Due

- If this was an ordinary annuity, its FV would be:

$$\begin{aligned} FV_A &= 100\,(1+0.1)^2 + 100\,(1+0.1) + 100 \\ &= 331 \end{aligned}$$

- If we multiple FV_A by (1+ *i*), we have

$$\begin{aligned} FV_A\,(1+i) &= 100(1+0.1)^3 + 100(1+0.1)^2 + 100(1+i) \\ &= 364.1 \end{aligned}$$

- That is, all payments would occur one period sooner → the FV of an annuity due!

PV and FV of Annuity Due vs. Ordinary Annuity

We DON'T need separate formula for PV and FV of **Annuity Dues** because

- PV of annuity due:

  = (PV of ordinary annuity) (1+***i***)

- FV of annuity due:

  = (FV of ordinary annuity) (1+***i***)

The PV and the FV of an annuity due are those of an ordinary annuity with the same cash flows compounded for one period.

Effective Annual Rate (EAR) or Effective Percentage (EFF%)

- The "finance charge rate" of a typical credit card in HK is around 24% APR. This quoted rate is a *nominal rate*.
- Since interest (or finance) charges on the account balance is computed on a monthly basis, the *periodic rate* is 24%/12 = 2% per month.
- What is the EFFECTIVE annual rate of the finance charge on the credit card?

Effective Annual Rate

- Effective Annual Rate (EAR) is the annual rate at which a PV will grow to the same FV as the nominal rate under multi-period compounding.
- That is:

$$\mathrm{PV}(1+\mathrm{EAR}) = \mathrm{PV}\left(1+\frac{i_{Nom}}{m}\right)^m$$

$$\mathrm{EAR} = \left(1+\frac{i_{Nom}}{m}\right)^m - 1$$

where *m* is the number of periods per year.

What is the EAR of a 24% nominal rate, compounded monthly?

- Back to our credit card finance charge example:

$$\begin{aligned}
\text{EAR} &= \left(1 + \frac{i_{Nom}}{m}\right)^{m} - 1 \\
&= \left(1 + \frac{0.24}{12}\right)^{12} - 1 \\
&= 1.02^{12} - 1 \\
&= 0.2682 \\
&= 26.82\%
\end{aligned}$$

Why Compute the EAR or EFF%

- Different kinds of investments make payments at different intervals – some make monthly payments while others may make quarterly payments.
- In order to compare the rates of return of investments with different payment intervals we need to put them on EFF% basis.

EAR (or EFF%) of 10% nominal rate

EAR_{Annual} = **10.00%.**

$EAR_{Qtr} = (1 + 0.10/4)^4 - 1$ = **10.38%.**

$EAR_{Mth} = (1 + 0.10/12)^{12} - 1$ = **10.47%.**

$EAR_{D(360)} = (1 + 0.10/360)^{360} - 1$ = **10.52%.**

More frequent compounding → higher EFF%

Exercise: Retirement Saving Example

- Consider the example in slide #70

  Suppose you open a saving account with $400 and invest $400 at the end of each month starting the following month for the next 30 years at 12%. How much will you have **at the end** of year 30?